

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 9

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Homework 9.1 (True or false?). Prove the following statements or give a counterexample.

- a. The image of a simply connected domain under a nonconstant holomorphic function is again a simply connected domain.
- b. The image of a simply connected domain under an injective holomorphic function is again a simply connected domain.
- c. The complex plane is not biholomorphically equivalent to any simply connected domain $G \subsetneq \mathbb{C}$.
- d. The set $\mathbb{C} \setminus B_r(z_0)$, where $z_0 \in \mathbb{C}$ and $r > 0$, is simply connected.

Homework 9.2 (Schwarz lemma on simply connected domains). Let $G \subsetneq \mathbb{C}$ be a simply connected domain. Given $a \in G$ we denote by $\text{Hol}_a(G)$ the set of holomorphic functions $f: G \rightarrow G$ such that $f(a) = a$. Show $|f'(a)| \leq 1$ for every $f \in \text{Hol}_a(G)$. Moreover, show $f \in \text{Hol}_a(G)$ is bijective if and only if $|f'(a)| = 1$ ¹.

Homework 9.3 (Singularities of injective holomorphic maps). Let $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic and injective, where $U \subset \mathbb{C}$ is open. Prove either z_0 is a removable singularity and the continuous extension to z_0 is still injective or z_0 is a pole of first order².

Homework 9.4 (Rigidity of biholomorphic maps*). a. Let $f: B_1(0) \rightarrow B_1(0)$ be a biholomorphic map which satisfies $f(a) = a$ and $f(b) = b$ for two distinct points $a, b \in B_1(0)$. Show f is the identity map on $B_1(0)$.
b. Show $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and injective if and only if f is of the form $f(z) = az + b$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$ ³.
c. Show $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$ is holomorphic and injective if and only if f is of the form $f(z) = az$ or $f(z) = a/z$ for some $a \in \mathbb{C} \setminus \{0\}$.
d. Let $G \subsetneq \mathbb{C}$ be a simply connected domain and let $f \in \text{Hol}_a(G)$ be biholomorphic, where $a \in G$ (cf. Homework 9.2). Show $f'(a) \in (0, \infty)$ implies f is the identity map on G .

Homework 9.5 (Examples of the Riemann mapping theorem). In this exercise we build biholomorphic maps $f: G \rightarrow B_1(0)$ for some special sets $G \subset \mathbb{C}$.

- a. Show the assignment $z \mapsto (z - i)(z + i)^{-1}$ is biholomorphic from the upper half-plane $\mathbb{H}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ to $B_1(0)$.
- b. Find a biholomorphic map $f: \mathbb{C} \setminus (-\infty, 0] \rightarrow B_1(0)$.

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¹**Hint.** Use the function given by the Riemann mapping theorem.

²**Hint.** Rule out an essential singularity using Picard's great theorem.

³**Hint.** Apply Homework 9.3.